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COMPUTATION OF COMPLEX FLUID FLOWS USING AN ADAPTIVE GRID METHOD

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SUMMARY

Recently the concept of adaptive grid computation has received much attention in the computational fluid dynamics research community. This paper continues the previous efforts of multiple one-dimensional procedures in developing and asessing the ideas of adaptive grid computation. The focus points here are the issue of numerical stability induced by the grid distribution and the accuracy comparison with previously reported work. Two two-dimensional problems with complicated characteristics—namely, flow in a channel with a sudden expansion and natural convection in an enclosed square cavity—are used to demonstrate some salient features of the adaptive grid method. For the channel flow, by appropriate distribution of the grid points the numerical algorithm can more effectively dampen out the instabilities, especially those related to artificial boundary treatments, and hence can converge to a steady-state solution more rapidly. For a more accurate finite difference operator, which contains less undesirable numerical diffusion, the present adaptive grid method can yield a steady-state and convergent solution, while uniform grids produce non-convergent and numerically oscillating solutions. Furthermore, the grid distribution resulting from the adaptive procedure is very responsive to the different characteristics of laminar and turbulent flows. For the problem of natural convection, a combination of a multiple one-dimensional adaptive procedure and a variational formulation is found very useful. Comparisons of the solutions on uniform and adaptive grids with the reported benchmark calculations demonstrate the important role that the adaptive grid computation can play in resolving complicated flow characteristics.

KEY WORDS Adaptive grid computation High Reynolds number flow

INTRODUCTION

The concept of adaptive grid solutions has recently received much attention. The most appealing aspect of the adaptive grid method is that the grid distribution can be adjusted in an intelligent way without resorting to *a priori* knowledge and/or the intuition of the user, and hence one can reduce the size of the grid system that is needed to yield an accurate solution. This was exemplified in a study conducted by De Vahl Davis and Jones.[1] In comparing the various numerical methods submitted by many individuals for calculating the natural convection in a square cavity, they found that, to their surprise, the use of a non-uniform grid distribution did not, on the whole, yield better numerical accuracy than that of a uniform grid distribution. This finding demonstrates that, while a denser distribution of mesh points in 'suitably chosen' locations should lead to improved accuracy, how to choose such suitable locations and the effects of the consequently coarsened grid distribution elsewhere on the numerical accuracy must be considered carefully.

A number of techniques for constructing adaptive grids for use in solving partial differential equations have been comprehensively surveyed by Anderson[2] and Thompson.[3] Russell and

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Christiansen[4] noted that all adaptive grid methods essentially attempt to equidistribute some weighting function $w(x)$ of the solution, i.e.,

$$\int_{x_i}^{x_{i+1}} w(x)\,\mathrm{d}x = \text{constant}$$

for a one-dimensional problem. For multi-dimensional Navier–Stokes flow the effective application of the adaptive grid method still awaits more research to investigate questions such as the different characteristics of the dependent varibles in a coupled system of equations, the non-linear behaviour of the flow and the extra complexities introduced by the non-regular flow configurations. An adaptive grid method based on the concept of equidistribution has been developed by the author as a multi-dimensional procedure and has shown interesting results.[5] The basic idea developed in Reference 5 was that for the Navier–Stokes equations the various dependent variables can have different characteristics depending on the flow configuration. For example, for flow through a channel with low Mach number, the no-slip restraint for the velocity on the solid surface combined with the relatively large Reynolds number can produce a highly non-uniform variation of the velocity profile along a specific direction. On the other hand, since the Poisson equation governs the pressure distribution and since a totally different kind of restraint is imposed on the pressure field by the solid surface, the pressure profile not only has a smoother variation but usually changes along different directions from the velocity profile. In Reference 5 it was demonstrated that, as a result of these fundamental differences between the structures of the pressure field and the velocity field, an adaptive grid strategy that employs multiple one-dimensional grid adjustment procedures using different weighting functions for different directions is useful.

In a follow-up study, the author demonstrates that one advantage of this *a posteriori* multiple one-dimensional adaptive grid method is its flexibility and ease in adding grid points along the co-ordinates if desired.[6] It was shown that in the multi-stage adaptive grid procedure the resulting grid system is already close to optimum after a few stages of adaption. Furthermore, as the adaptive readjustment of the grid distribution proceeds from the initial grid system, not only is the overall error reduced but the error distribution appears more uniform. These findings are consistent with the simple model problem analyses reported in Reference 7.

As to the skewness of the mesh that might be inroduced by multiple one-dimensional procedures, Braaten and Shyy[8] found, in a somewhat different context, that in practice considerable skewness of the individual meshes does not affect the overall numerical accuracy in any significant way. Thompson *et al.*[9] noted that, based on a local analysis of the Taylor series expansion, for a mesh with angles no smaller than 45° between the intersecting co-ordinate lines there are virtually no adverse effects of mesh skewness on the numerical accuracy. The study conducted in Reference 8 not only supports the assessment of Thompson *et al.* but also indicates that the effects of more excessive local skewness on the overall numerical accuracy of the Navier–Stokes equations are quite tolerable.

In the present study the issue of numerical stability of the adaptive grid method will be investigated. In particular it will be demonstrated that by appropriately distributing the grid points the numerical algorithm can more effectively dampen out the instabilities, especially those related to artificial downstream boundary treatments and higher-order finite difference operators, and hence can more rapidly converge to a steady-state solution. Both laminar and turbulent flows will be investigated, and the responsiveness of the grid distribution with respect to the Reynolds number will be studied. Next, the problem of two-dimensional natural convection in an upright square cavity will be considered. It will be shown that an adaptive procedure combining a multiple one-dimensional procedure and a variational formulation can also be effectively applied to this problem to yield accurate solutions.

NUMERICAL ALGORITHM

The governing equations together with the co-ordinate transformations have been discussed previously[5] and will not be elaborated here. Briefly, a staggered grid system[10] is adopted, and the implementational details in the context of a curvilinear co-ordinate system can be found in References 8 and 11. The momentum equations are first solved to obtain the velocity components with an assumed pressure field. After solving the momentum equations, the contravariant velocity components are calculated from the velocity field. If the calculated velocity field does not satisfy the continuity equation, the pressure distribution and velocity field are corrected accordingly.

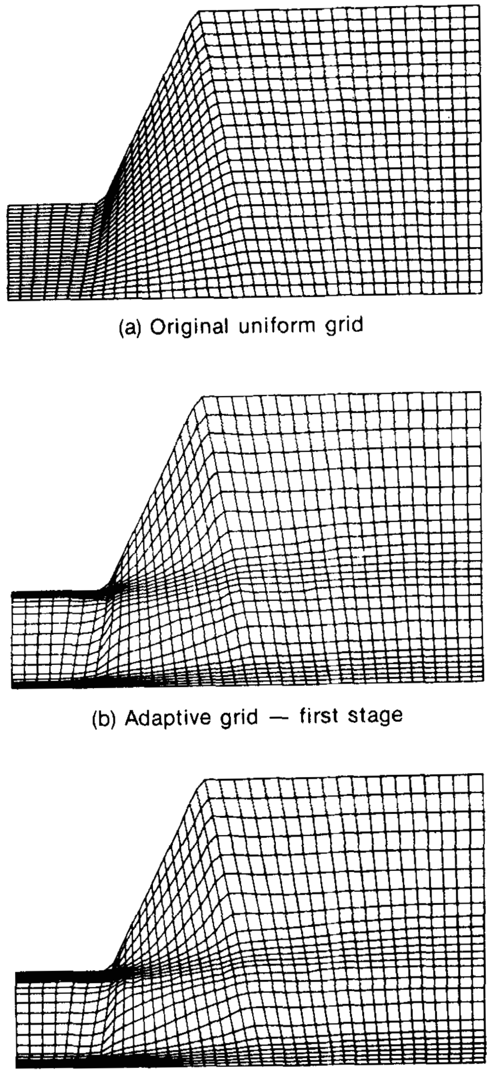
For the numerical treatment of the downstream open boundary, the formula suggested in Reference 12 is adopted, i.e., $\phi_\xi = 0$ is taken for u- and v-velocity components, where ξ-lines are along the streamwise direction. The static pressure does not require any prescription of the boundary condition owing to the use of the staggered grid system.[10]

In terms of the finite difference operators approximating the various terms in the Navier–Stokes equations, all but the convection terms are discretized by using the standard second-order central differencing scheme. For the convection terms various upwind-based schemes are also available, as investigated in References 13–15. In the present numerical framework the so-called hybrid scheme,[10] a selective use of the first-order upwind or second-order central differencing scheme, dependent upon whether the local cell Reynolds number is larger than two or not, is always adopted at the nodes next to all boundaries for all upwind-based schemes discussed in References 13 and 14 to avoid the need for extrapolation and to stabilize the numerical procedure. This procedure is able to yield satisfactory results which are consistent with those reported by Blottner.[16] The detailed numerical implementation for these schemes can be found in Reference 17. In the illustration of the numerical results, both the second-order upwind scheme and the hybrid scheme will be adopted.

EFFECTS OF GRID DISTRIBUTION ON CONVERGENCE RATE

The first flow configuration under consideration is that of a two-dimensional planar channel with a one-sided sudden expansion. The flow is incompressible and laminar, and the inlet velocity is taken as a plug profile with Reynolds number, based on the inlet height, equal to 250. The original uniform grid with 41×26 nodes is shown in Figure 1(a). It is noted that for the flow in the present configuration with high Reynolds number a large recirculating eddy is formed in the expansion region which can run across the downstream boundary of the computational domain, which is very different from the studies conducted in References 5 and 6. Consequently, the interaction of the boundary treatments and the numerical algorithm for the interior domain can affect the convergence of steady-state solution.[10,18] Nevertheless, it is demonstrated in Reference 18 that for the flow considered here the steady-state solutions can be obtained numerically and the flow characteristics prove to be very insensitive to the position of the downstream boundary.

Figures 1(a)–(c) show the grid distributions from the original and two stages of the adaptive procedure. In the present notation each adaptive stage constitutes (i) the creation of a new adaptive grid system based on the given grid system and the numerical solution obtained from that grid system, and (ii) the flow field calculated on this adaptive grid system. Thus the present adaptive grid method is in the category of a sequential procedure. In this manner, although the number of



(a) Original uniform grid

(b) Adaptive grid — first stage

(c) Adaptive grid — second stage

Figure 1. Adaptive grid procedure for flow in channel with one-sided expansion (hybrid scheme, $Re = 250$)

adaptive stages can in principle continue indefinitely, in practice it only needs a few stages to reach an optimum grid distribution, as will be demonstrated. Here, the weighting function along the η-line used in the adaptive procedure is

$$w = 1 + |u_\eta| + |v_\eta|.$$

The subscript designates the derivative of the variable along the η-family of co-ordinate lines (cross-stream direction). In the context of an equidistribution constraint, this weighting function more or less produces a constant arc length in the solution curve across each mesh spacing. For the weighting function along the ξ-line (streamwise direction), since this particular geometry only produces a very moderate pressure gradient along the axial direction, it was found that $w = 1$ is satisfactory; i.e., the mesh spacing is equally distributed along the ξ-line.

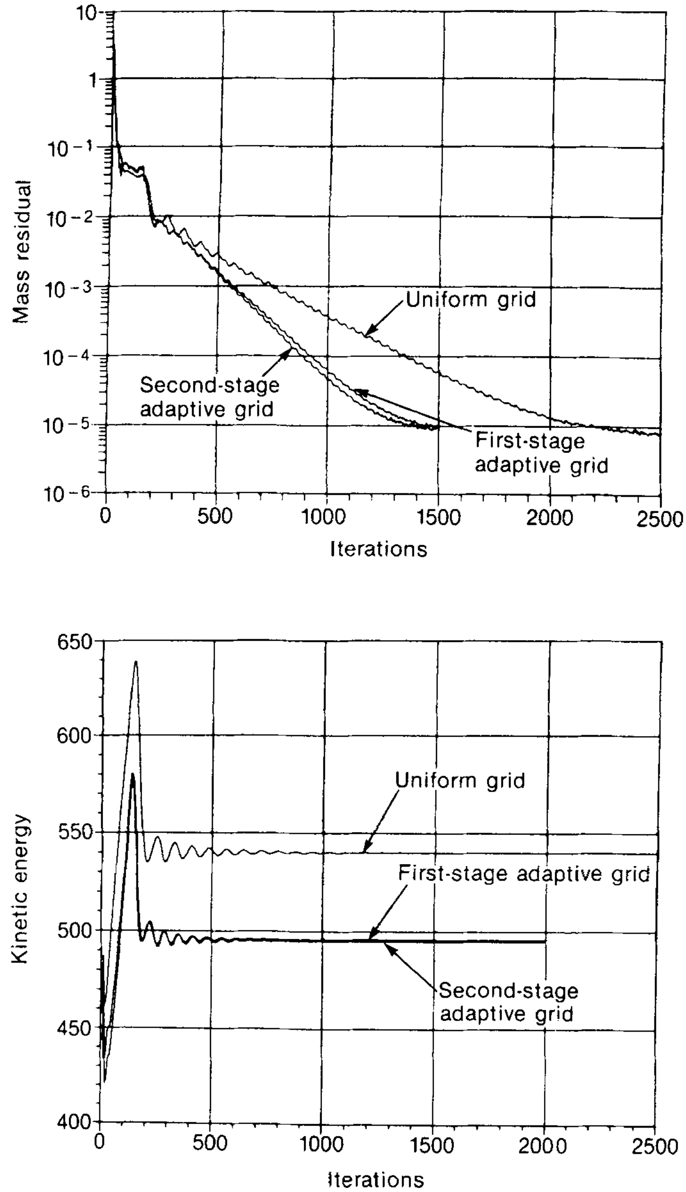
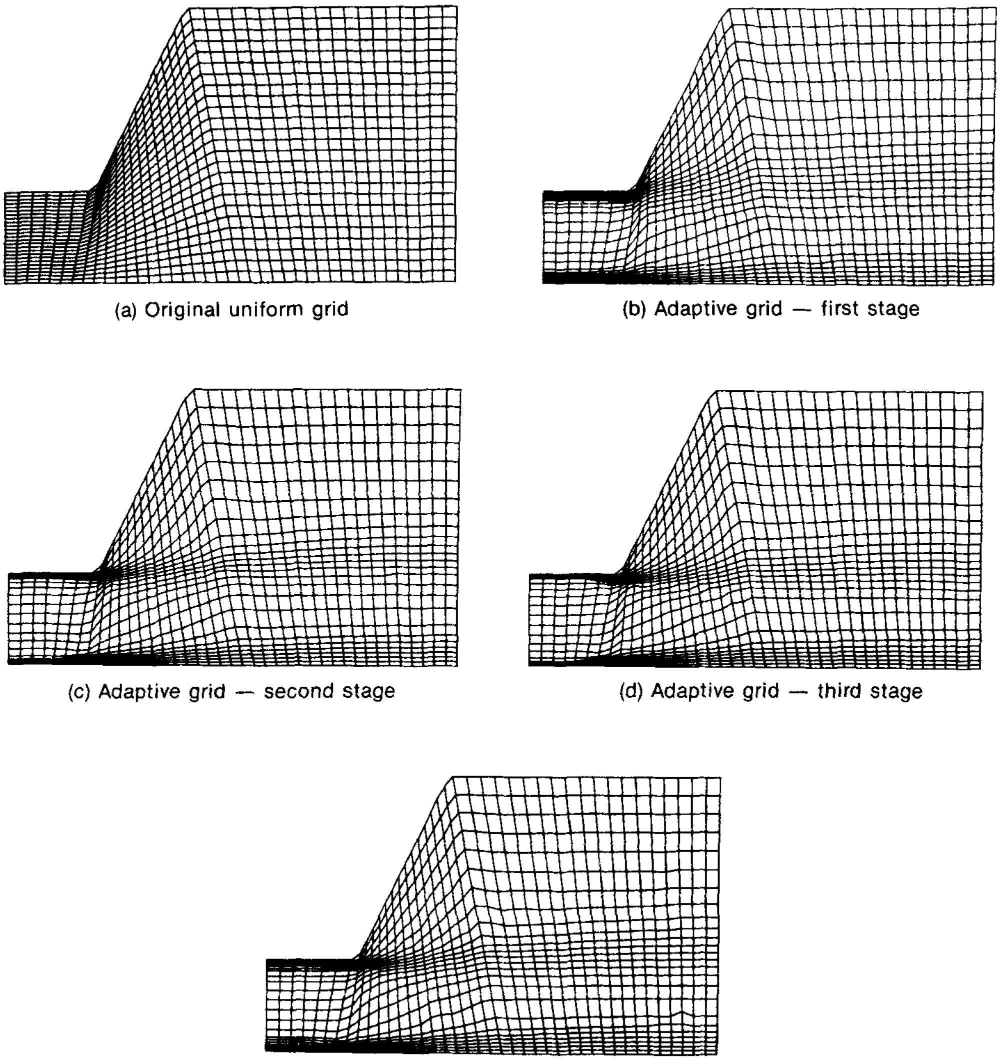


Figure 2. Effects of grid distribution on convergence rate (hybrid scheme, $Re = 250$)

Two finite difference operators approximating the convection terms are investigated here: the so-called hybrid scheme[10] and the second-order upwind scheme.[13] The adaptive grid systems together with their influences on the convergence rates (based on the same initial guesses, i.e., all the cases have been run for the same initial condition) for the hybrid scheme are shown in Figures 1 and 2. The mass residual and kinetic energy are defined respectively as the absolute sum of the mass flux unbalances in all the computational cells and the unweighted sum of the kinetic energies at all the nodal points. The kinetic energy indicator does not represent the integrated overall kinetic energy carried by the fluid, but it has been found to be a very useful



(a) Original uniform grid

(b) Adaptive grid — first stage

(c) Adaptive grid — second stage

(d) Adaptive grid — third stage

(e) Adaptive grid — fourth stage

Figure 3(a). Adaptive grid procedure for flow in channel with one-sided expansion (second-order upwind scheme, $Re = 250$)

tool for assessing the convergence of the numerical solution. It has been seen that for this particular problem with the hybrid scheme, one stage of adaptive grid is sufficient. Besides resolving the flow scales in a better way, the number of iterations needed with the adaptive grid system is about 40% less than with the uniform grid. The adaptive process with the second-order upwind scheme is shown in Figures 3 and 4. It is striking to observe that initially, on the uniform grid system, the numerical instabilities introduced by the numerical treatment along the downstream open boundary cause the mass residual to oscillate at a fixed level without dropping off. After one stage of adaptive computing, however, the grid system has an adjusted distribution which shows finer resolution in the free shear layer; the flow gradients are higher and can more effectively dampen out the disturbances, while the convergence rate is essentially comparable with that of the hybrid scheme. This result once more demonstrates that, although the convection

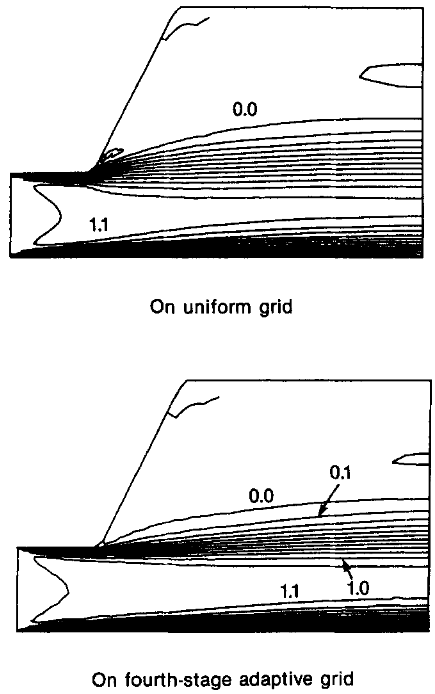


Figure 3(b). Calculated constant u-velocity contours on uniform and adaptive grids (second-order upwind scheme, $Re = 250$)

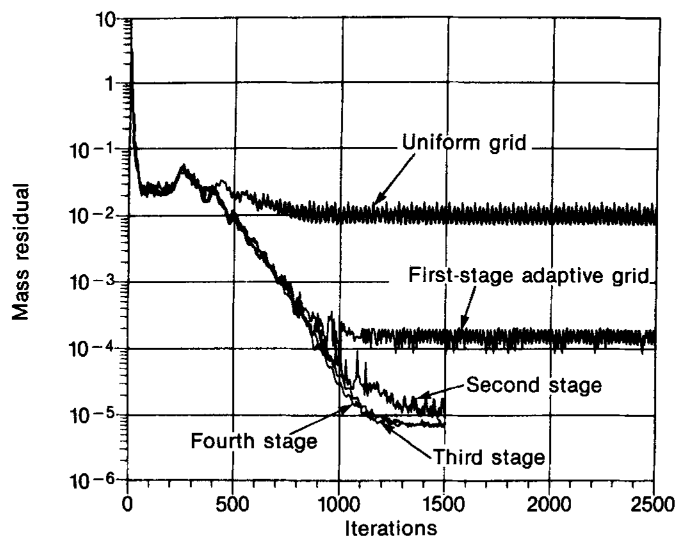


Figure 4. Effects of grid distribution on convergence rate (second-order upwind scheme, $Re = 250$)

term approximations are very important in determining the numerical accuracy, the distribution of the grid points should receive equal attention. The calculated u-velocity contours on uniform and adaptive grids are shown in Figure 3(b).

A turbulent flow with the inlet Reynolds number equal to $2{\cdot}5 \times 10^5$ has also been studied for the

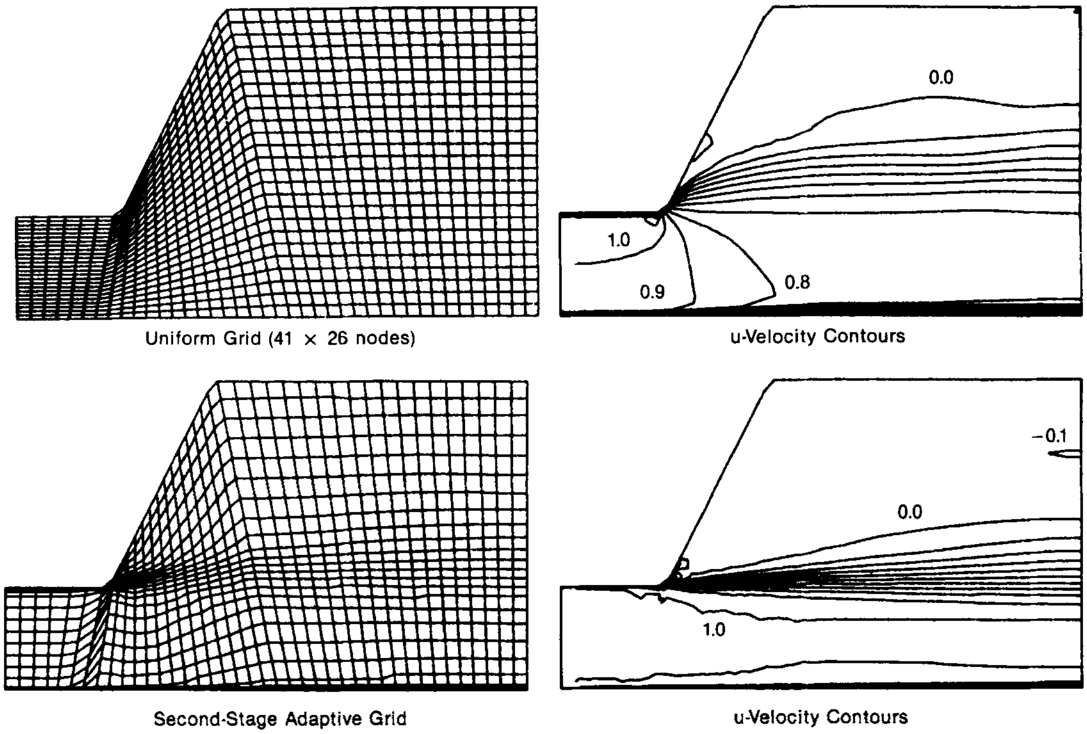


Figure 5(a). Grid systems and calculated solutions with second-order upwind scheme, $Re = 2{\cdot}5 \times 10^5$

same geometrical configuration. The standard k–ε two-equation model with the wall function treatment along the solid wall is used; all other boundary conditions remain the same as for the laminar flow cases. Figure 5 shows the grid system, the calculated solutions and the convergence path for the computation using the second-order upwind scheme. It is seen that the characteristics of the convergence paths for both the laminar and turbulent flows are similar; the adaptive grid is needed to resolve the flow features better and hence to dampen out the numerical instabilities. On the other hand, the velocity gradients in the wall and free shear flow regions are steeper for the turbulent flows. These characteristics are well reflected in the adaptive grid systems, which show that for the turbulent flow calculations the grid densities are higher in the regions of high velocity gradient.

NATURAL CONVECTION FLOW

The problem of natural convection of Navier–Stokes flow with the Boussinesq approximation in an upright two-dimensional square cavity is entirely prescribed by two dimensionless parameters, the Rayleigh number Ra and the Prandtl number Pr. The description of the problem is shown in Figure 6. Much numerical work has been reported in the literature, e.g. Reference 19, for flows up to $Ra = 10^6$. For laminar flow with $Ra > 10^6$ stable and accurate numerical solutions are difficult to obtain. The work by Le Quere and De Roquefort,[20] who use a semi-implicit spectral method, is one of the few reported studies for $Ra = 10^7$ and higher.

In the present work the adaptive grid method is applied to laminar natural convection flow in a square cavity with $Ra = 10^5$–10^7 and $Pr = 0{\cdot}71$. For the problem considered here the same

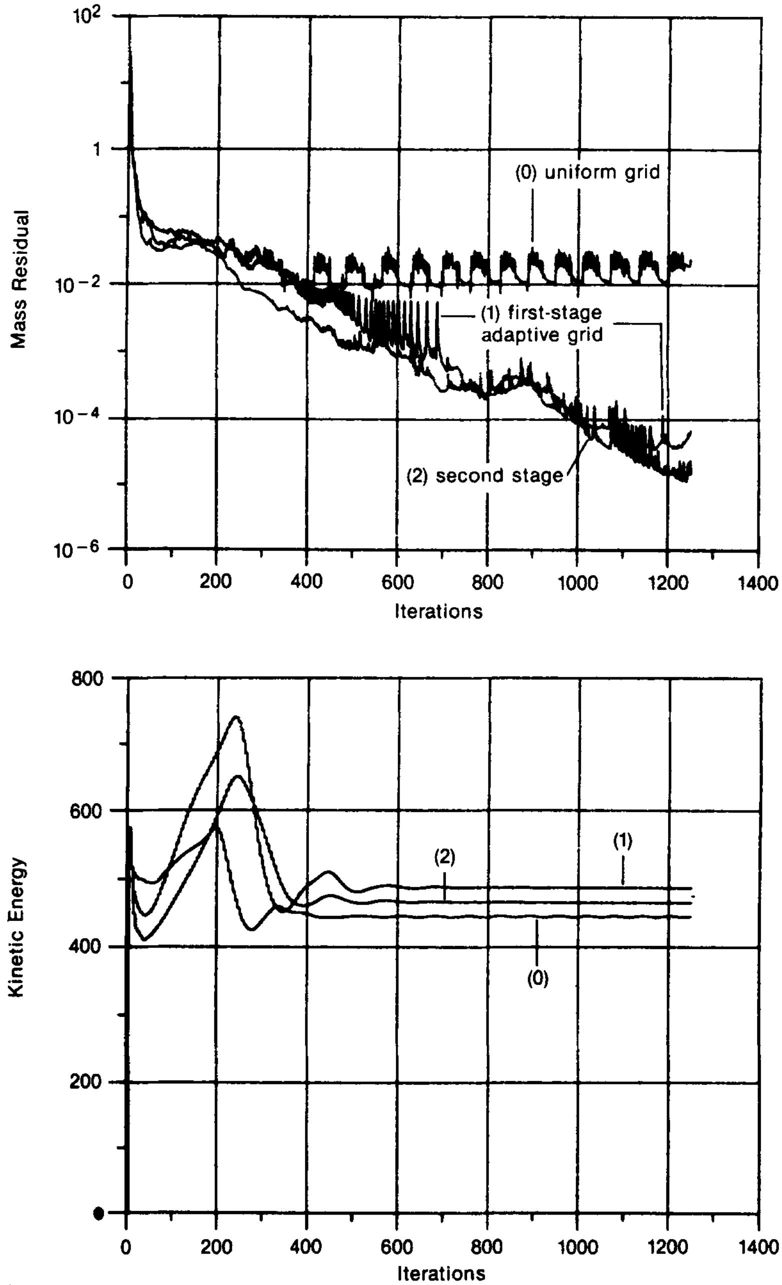


Figure 5(b). Effects of grid system on convergence rate for flow in channel (41 × 26 grid) with second-order upwind scheme, $Re = 2{\cdot}5 \times 10^5$

weighting function is applied to both ξ- and η-co-ordinate lines, since the flow is contained within an enclosed cavity and the no-slip constraint is dominant along boundaries. The weighting function used in this study is

$$w(s) = a + |u_s| + |v_s|,$$

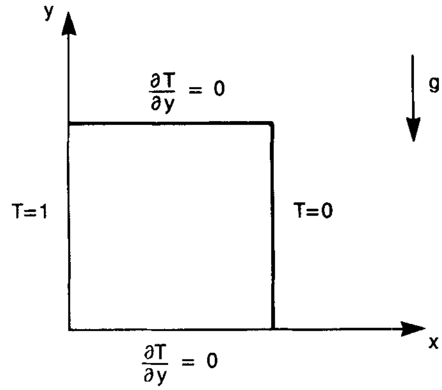


Figure 6. Schematic of natural convection problem in square cavity

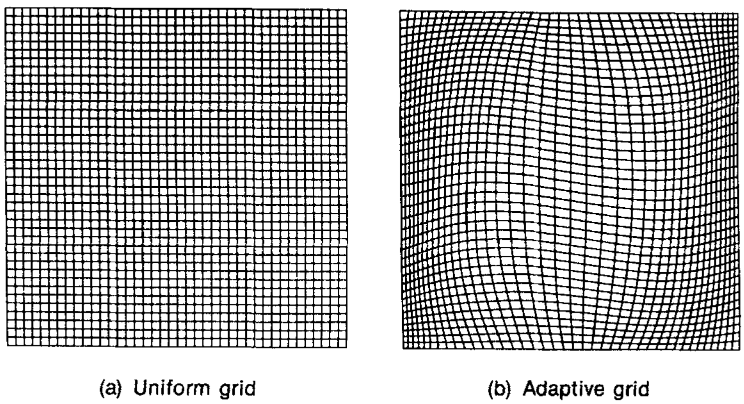
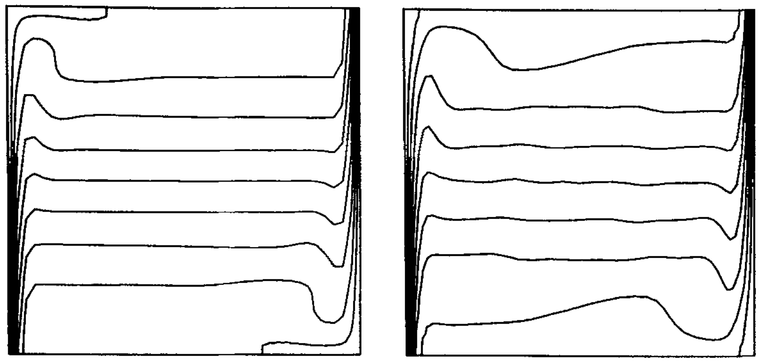


Figure 7. Grid systems of the natural convection flow in a square cavity (41 × 41 nodes, $Ra = 10^7$)

where a is a constant taken as 5 here to smooth out the grid distribution[5] and u_s and v_s are the gradients of the velocities (normalized by the maximum values) along the ξ- and η-co-ordinate lines. The adaptive procedure is conducted after the solution on the original grid is obtained and can proceed iteratively.

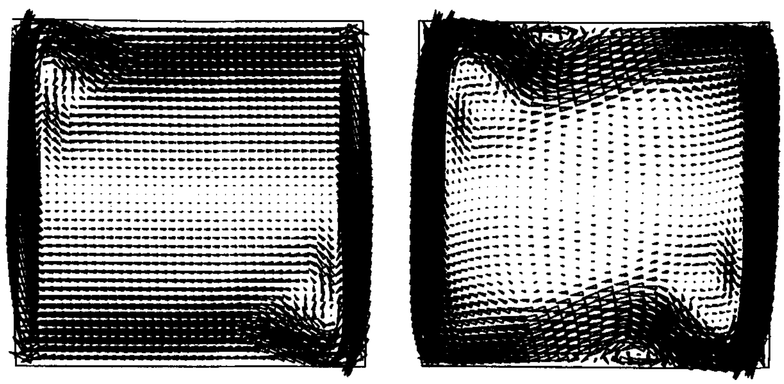
Since the solution is skew-symmetric with respect to the centre point of the cavity, the resulting mesh distribution of the adaptive grid procedure is more skewed than desirable. To maintain the numerical stability of the solution algorithm, it is found very useful to make the interior grid distribution smoother and more orthogonal while fixing the boundary points obtained from the adaptive procedure. This post-processing strategy yields a better compromise between the solution adaption and the grid smoothness. The variational principle developed by Brackbill and Saltzmann[21] was applied to perform this task, with the smoothness and the orthogonality being given equal emphasis. This procedure has been utilized previously[17] as an aid to generating a grid system which can balance the smoothness and orthogonality.

A uniform grid system with 41 × 41 nodal points was used as the original grid system. The original and adaptive grid systems are shown in Figure 7. The adaptive grid system is based on the numerical solution obtained on the uniform grid. A new calculation was then conducted on the adaptive grid system to yield an adaptive grid solution. Compared with the original grid system, the adaptive grid system concentrates the grid distribution in the corner and wall regions in a non-uniform manner. Figure 8 compares the temperature contours for the results calculated on the two



(a) Uniform grid solution (b) Adaptive grid solution

Figure 8. Comparison of constant-temperature contours between two grid solutions (41 × 41 grid, $Ra = 10^7$)

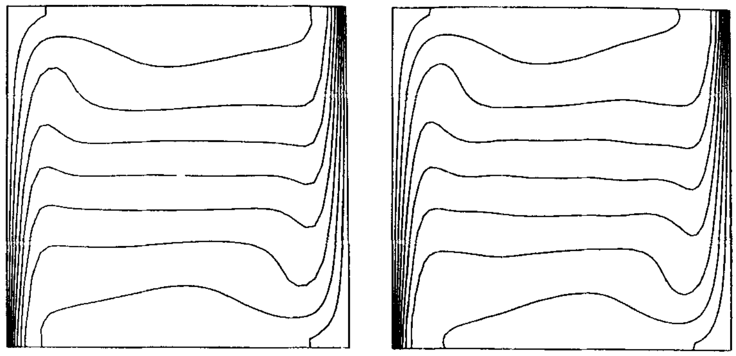


(a) Uniform grid solution ($|v|_{max} = 509$) (b) Adaptive grid solution ($|v|_{max} = 682$)

Figure 9. Comparison of velocity fields between two grid solutions (41 × 41 grid, $Ra = 10^7$)

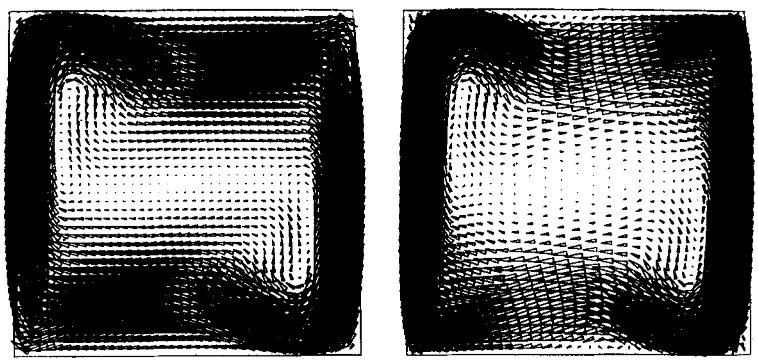
grid systems. The adaptive solution shows sharper peaks close to the side walls and has a more curved temperature distribution in the top and bottom regions, which is consistent with the results reported in Reference 20. Figure 9 compares the velocity vectors for the solutions obtained on the two grid systems. The magnitudes of all the vectors have been scaled up to depict the fine flow structures. Noticeable differences in the flow characteristics are observed. The streamlines of the adaptive grid solution are again closer to those shown in Reference 20. The adaptive grid solution shows more complicated vortex cell structures within the bulk fluid; it also depicts extra recirculating bubbles in the top and bottom wall regions which are absent from the uniform grid solution. The computed maximum value of the v-velocity component on the adaptive grid system is 682, which agrees with that reported in Reference 20 to within 2·5%. In Reference 20, 65 × 65 uniform nodes were used in conjunction with the spectral method. On the other hand, the computed maximum value of the v-velocity component of the present study on the uniform grid is 509.

Figures 10 and 11 compare the temperature and velocity fields calculated on the uniform and adaptive grid systems for $Ra = 10^6$, while Figures 12 and 13 show the same comparison for $Ra = 10^5$. Table I compares the maximum values of the vertical velocities obtained from the present



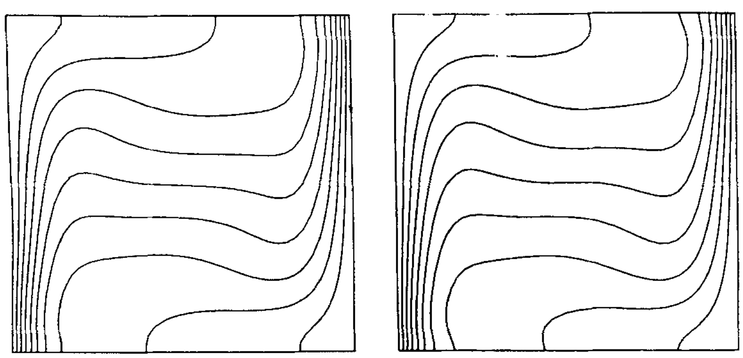
(a) Uniform grid solution (b) Adaptive grid solution

Figure 10. Comparison of constant-temperature contours between two grid solutions (41 × 41 grid, $Ra = 10^6$)



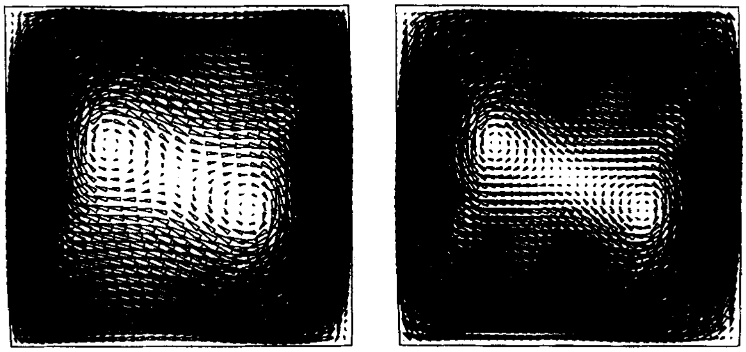
(a) Uniform grid solution ($|v|_{max} = 198.0$) (b) Adaptive grid solution ($|v|_{max} = 219.7$)

Figure 11. Comparison of velocity fields between two grid solutions (41 × 41 grid, $Ra = 10^6$)



(a) Uniform grid solution (b) Adaptive grid solution

Figure 12. Comparison of constant-temperature contours between two grid solutions (41 × 41 grid, $Ra = 10^5$)



(a) Uniform grid solution ($|v|_{max} = 66.70$)   (b) Adaptive grid solution ($|v|_{max} = 67.96$)

Figure 13. Comparison of velocity fields between two grid solutions (41 × 41 grid, $Ra = 10^5$)

work with three benchmark-type results reported in the literature[19,20,22] where care has been taken to ensure good accuracy of the numerical results. With the present size of grid system, both the uniform and adaptive grid results are quite accurate for $Ra = 10^5$, where the flow structure varies in a smooth manner and hence the uniform grids are adequate to use. As the Rayleigh number increases, the scales of flow vary more rapidly and so do the discrepancies between the uniform grid and adaptive grid solutions. On the other hand, the adaptive grid results for all cases are in good agreement with those reported in References 19, 20 and 22 to within the numerical uncertainties. These comparisons clearly demonstrate the important role that the adaptive grid computation can play in resolving complicated flow characteristics, especially when the appropriate grid distribution cannot be accurately determined beforehand.

As to the role of the pressure gradient, as shown in Reference 5, for the present expansion channel flow the variation of the pressure gradient is very moderate and hence is not influential in the weighting function. Reference 5 also shows that the inclusion of the pressure term in the weighting function is more influential for the contraction channel flow. For the natural convection cavity flow, since the dominant constraint is from the enclosed walls on all sides, the velocity gradient is again more influential. The point is that the multiple one-dimensional adaptive grid approach can be combined with various weighting functions whose most appropriate forms should be dependent upon the flow characteristics.

CONCLUSIONS

Based on the results reported here, the following observations can be made.

(1) It has been demonstrated that the adaptive grid solution can usefully improve the accuracy of the numerical solution with a fixed number of grid points.

(2) By adaptively adjusting the grid distribution according to the flow characteristics, the numerical algorithm can more effectively dampen out the instabilities and hence can converge to a steady-state solution more rapidly. For a more accurate finite difference operator, such as the second-order upwind scheme, which contains less undesirable numerical diffusion, the adaptive grid system can yield a steady-state and convergent solution, while uniform grids produce non-convergent and numerically oscillating

Table I. Comparison of calculated maximum vertical velocity for natural convection in a square cavity

Source*	Maximum vertical velocity
(I) $Ra = 10^4$	
1. DeVahl Davis[19]	19·62
2. Markatos and Pericleous[22]	19·44
3. Le Quere and De Roquefort[20]	19·63
4. Present work: 41 × 41 uniform grid	19·42
5. Present work: 41 × 41 adaptive grid	19·52
(II) $Ra = 10^5$	
1. DeVahl Davis[19]	68·59
2. Markatos and Pericleous[22]	69·08
3. Le Quere and De Roquefort[20]	68·65
4. Present work: 41 × 41 uniform grid	66·70
5. Present work: 41 × 41 adaptive grid	67·96
(III) $Ra = 10^6$	
1. DeVahl Davis[19]	219·36
2. Markatos and Pericleous[22]	221·8
3. Le Quere and De Roquefort[20]	220·57
4. Present work: 41 × 41 uniform grid	198·0
5. Present work: 41 × 41 adaptive grid	219·7
(IV) $Ra = 10^7$	
1. DeVahl Davis[19]	No data
2. Markatos and Pericleous[22]	No data
3. Le Quere and De Roquefort[20]	699·3
4. Present work: 41 × 41 uniform grid	590·5
5. Present work: 41 × 41 adaptive grid	682·0

* Remarks on various computational techniques:
[19] Second-order central difference scheme for convection terms.
Uniform 81 × 81 grid and Richardson extrapolation.
Claims to be accurate to better than 1%.
[22] First-order upwind scheme for convection terms.
Non-uniform 41 × 41 grid.
No mention of criteria for grid distribution.
[20] Spectral method with Chebyshe polynomials.
Uniform 65 × 65 grid.
Agreement to third decimal place between solutions on 33 × 33 and 65 × 65 grids for $Ra = 10^6$.

solutions. It is noted that this phenomenon is, in some way, uniquely related to the fact that for the present channel configuration, the recirculating flow passes across the open outflow boundary. In Reference 6, where the open boundary does not contain any recirculating flow, the differences in convergence rate among different grid systems are less noticeable.

(3) Both laminar and turbulent flows have been calculated, and the resulting adaptive grid systems reflect well the differences in the flow characteristics.

(4) Although the multiple one-dimensional adaptive procedure does not explicitly control the local mesh skewness, numerical evidence (References 5, 6, 8 and the present study) appears to suggest that the effects of local mesh skewness on the overall numerical accuracy are usually not significant.

(5) As to the choice of the finite difference operators approximating the convection terms, while

the different schemes have different intrinsic stability and accuracy characteristics,[13,14] it is obvious that a balanced compromise of a good finite difference operator and a good grid distribution is certainly a very desirable and workable approach.

Overall, it is clear that the adaptive grid solution can be very useful in improving both the convergence rate and the numerical accuracy. It appears that by devising the appropriate weighting function and adaptive procedures according to the requirements of the flow characteristics, such as for the cases shown in the present study, numerical solutions with desirable accuracy can be obtained for many problems without resorting to an extremely large grid system.

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